Definitions

Sets

set of all 2-column-combinations for a dice throw

set of all columns

set of temporary columns

set of open columns

set of closed columns

columns are either open or closed

temporary columns can be open or closed

Probabilities

probability of **hitting temporary** columns,  
 that are also open

probability of **missing temporary** columns,  
 that are also open

probability of ending turn without miss

Gains & expected gains

current gain (variable)

gain in case of

gain in case of

expected gain for dice throw

expected gain for entire turn

Analysis

Definition of gain

As the goal of the game is to reach the top of a column, we define gain as a value proportional to reaching this goal. Thus gain is expressed as a fraction of reaching the top of a column. Reaching the 3rd step of a 5-step-column is represented by a gain of 3/5.

Expected gain for entire turn

We will first try to optimize the expected gain for an infinitely large board with free movement. With this assumption, lines can't be closed during the turn and we can end the turn at any moment.

Now we will cast the dice as long as the expected value of a dice throw is positive:

For the in-between case, we conveniently define the break-even gain :

At the break-even gain, both decisions are equivalent in terms of expected gain.

Expected gain for 3-column-combination

The expected value depends on the probabilities of hitting () and missing (), scaled by the gain in either case:

is constant and depends on the average intersection of dice-combinations with the temporary columns . The current gain is a sum of the gains in the temporary columns. The higher the current gain , the lower the expected value, as we risk more. We are interested in the value of , at which we should end our turn. Implementing the condition of :

Using the definitions and :

Now we now, when to stop casting the dice. This brings us one step closer to calculating the expected value for a 3-column-combination. We will cast the dice, until exceeds and save the gain , in case we succeeded. Yet, at which gain do we actually end the turn? After all, the gain is not continuous, but quantized. Here, we will make another simplifying assumption, that we will on average overshoot the break-even gain by half of the quantized step-size . Thus, we expect to end our turn at the gain :

Finally, the expected gain of the entire turn is the product of the gain and the probability, to reach this goal (:

depends on the current gain , as our probability of success increases, the closer our current gain gets to the intended gain . The difference between the current gain and the goal can be expressed as a sum of quantized steps:

Thus, is the number of single gains necessary, to reach . For each single gain , the probability of success is . Therefore, the probability can be expressed as:

And finally for the expected gain of the entire turn:

with

With this formula, we can calculate the expected gain for the turn , if and are known and constant. This is the case, once we have chosen 3 columns.

Expected gain for 2-column-combination

Once we choose a 2-column-combination, for the following dice throw, there are two types of outcomes:

1. we choose a 3-column-combination
2. we choose to remain with 2 columns for at least another dice throw

For the first case, we already now, how to calculate the expected value. For the second case, the probabilities for the following throws won’t change, the only variable is the gain.

In theory, we have various gains, at which we decide to choose a 3-column-combination, depending on the number of times, that we managed to remain with 2 columns. We will approximate these gains by an effective expected gain . The effective expected gain is the average gain of remaining in 2 columns.

Now we want to express the infinite sum differently. Renaming we get a series :

This looks very similar to the geometric series :

Differentiating we get:

And finally multiplying by :

Thus, we can express our series in terms of the geometric series:

This simplifies our equation significantly:

Which interestingly looks very similar to the break-even gain .

is the probability of reaching a certain 3-column-combination.

Recursively, this formula also applies for combinations with less than 2 columns.